TU WIEN

Probably Approximately Global Robustness Certification

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Problem Setting: Certification of Neural Network Robustness

Black-Box ML in Critical Applications: Why not?

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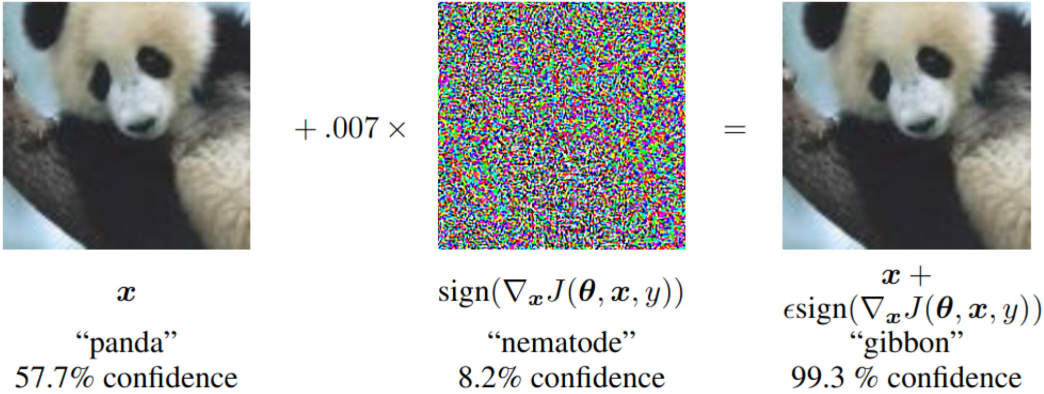
Goodfellow et al (2015)



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Black-Box ML in Critical Applications: Why not?

Athalye et al (2018)

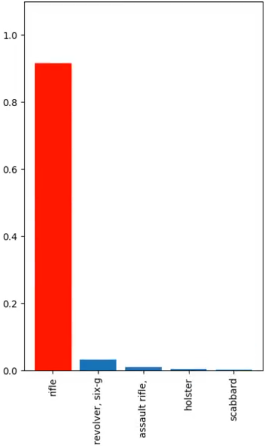

Image Source: Youtube Video

Black-Box ML in Critical Applications: Why not?

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Adversarial examples are a security risk

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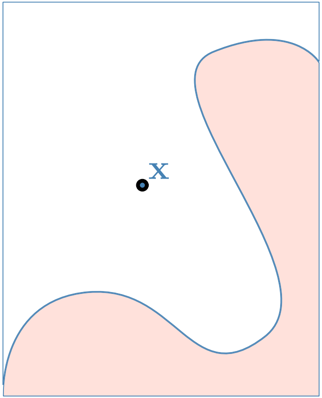

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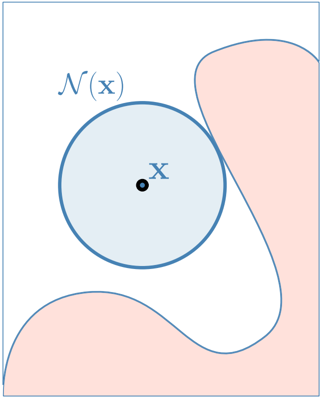

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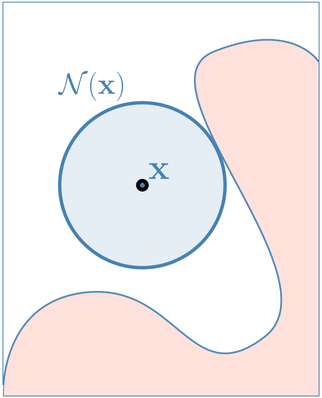
We focus on *certification* of robustness



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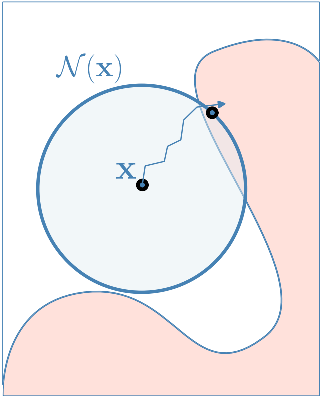

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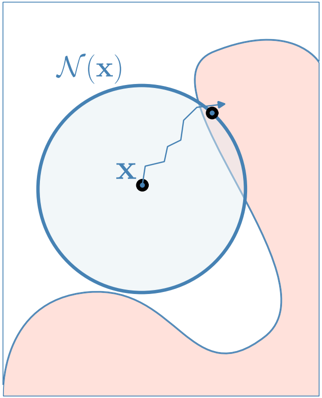
Where $\Pi(.)$ projects its argument back into $\mathcal{N}(x)$



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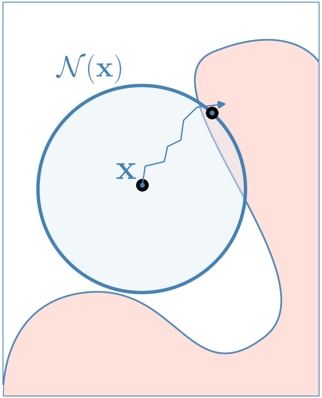
Good at finding adversaries... *but not exhaustive!*

(Finding adversarial examples is *hard* (Carlini and Wagner (2017)))



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- Results depend attack parameters
- Information gain about f is limited

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$$\forall \mathbf{x} \in \mathcal{X} : \forall \mathbf{x}' \in \mathcal{N}(\mathbf{x}) : \mathbf{class}(f(\mathbf{x})) = \mathbf{class}(f(\mathbf{x}')) \qquad (3)$$

Is too strict!

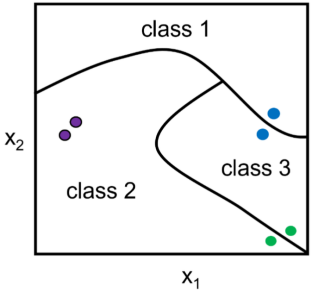

Image Source: Athavale et al (2024)

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We give the network the option to abstain and only consider *confident* predictions

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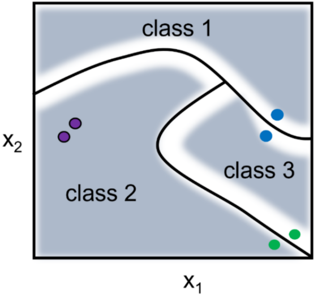

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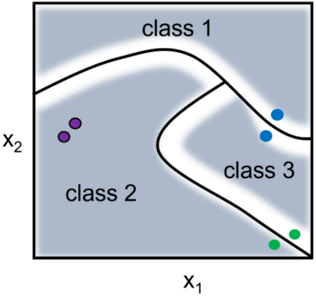

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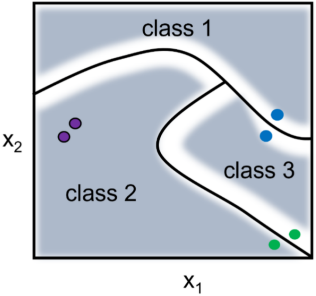
Very expensive, infeasible above 100s of neurons



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Our Objective:

- give sample based guarantees about global robustness
- Stay model-agnostic
- *Give specific robustness bounds for each prediction*

Background: Probabilistic Coverage Guarantees with Epsilon-nets

ϵ-Nets

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Definition (ϵ-Nets)

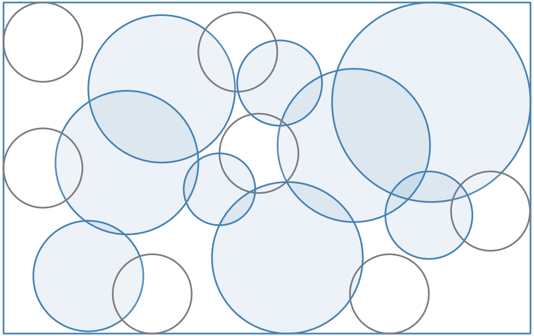
Given a range space $(\mathcal{X}, \mathcal{R})$ and a probability distribution $\mathcal{D}$, a finite set $N \subset \mathcal{X}$ is called an ϵ-*net*, iff N intersects each ϵ-probable $R \in \mathcal{R}$, i.e.,

$$\forall R \in \mathcal{R} : \Pr(R) \geq \epsilon \Rightarrow N \cap R \neq \emptyset \quad \Leftrightarrow \tag{5}$$

$$\forall R \in \mathcal{R} : N \cap R = \emptyset \Rightarrow \Pr(R) < \epsilon \tag{6}$$

ϵ-Nets: Example

We consider the range space $(\mathbb{R}^2, \mathcal{B})$, with $\mathcal{B}$ is some set of circles



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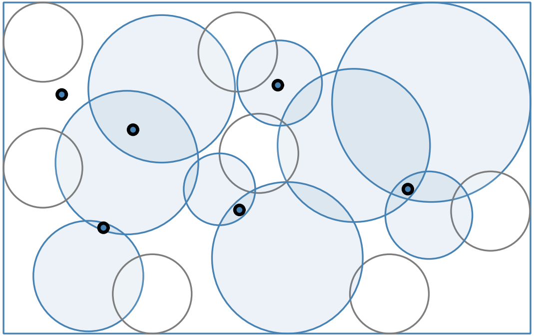
An ϵ-net intersects all likely enough circles



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Definition (VC-Dimension Vapnik and Chervonenkis (2015))

Let $(\mathcal{X}, \mathcal{R})$ be a range space. The Vapnik-Chervonenkis (VC) dimension d of $(\mathcal{X}, \mathcal{R})$ is the size of the largest set $S \subseteq \mathcal{X}$, such that

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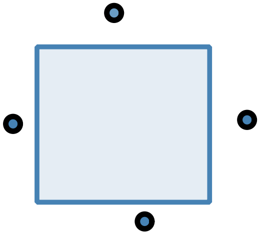

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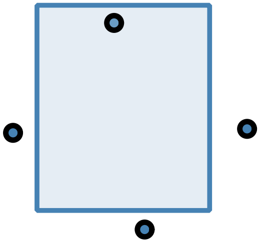

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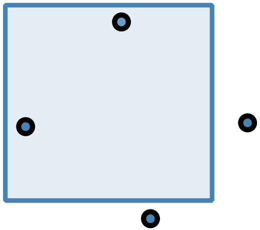

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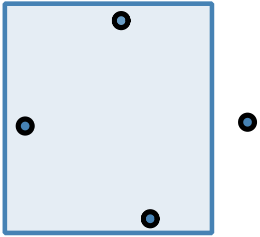

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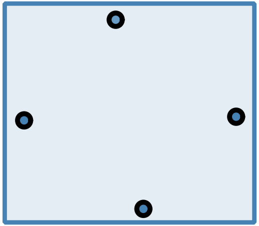

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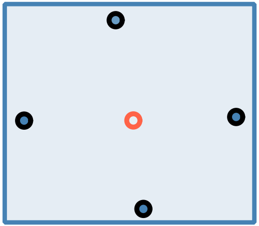

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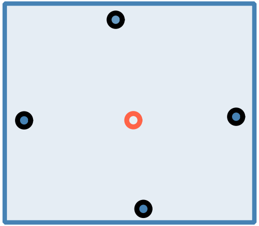

Well studied for common hypothesis spaces

ϵ-Nets from iid Samples

Theorem (ϵ-nets from iid samples (Mitzenmacher and Upfal (2017)) **)**

Let $(\mathcal{X}, \mathcal{R})$ be a range-space with VC-dimension d and $\mathcal{D}$ be a probability distribution. For any $0 < \delta, \epsilon \leq \frac{1}{2}$, an iid sample N will be an ϵ-net with probability at least $1 - \delta$ iff

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We are interested in obtaining minimal samples of sufficient size, so we find $|N| = s$ with

$$s(\epsilon, \delta, d) = \min_{s \in \mathbb{N}} \left\{ s : s \geq \frac{2}{\epsilon}\left(\log\left(\frac{2}{\delta}\right) + d \log(2s)\right)\right\} \tag{9}$$

Distillation with probably approximately global coverage

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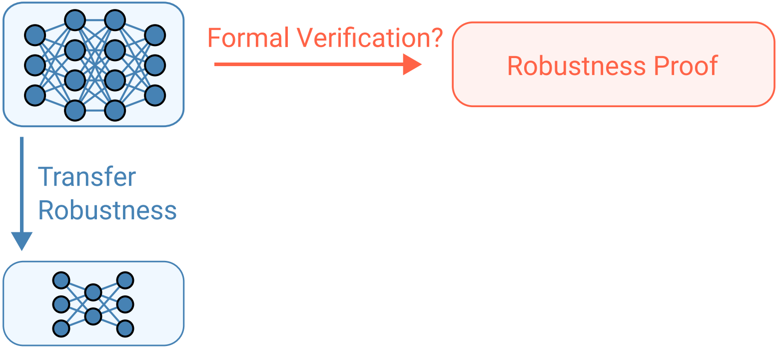

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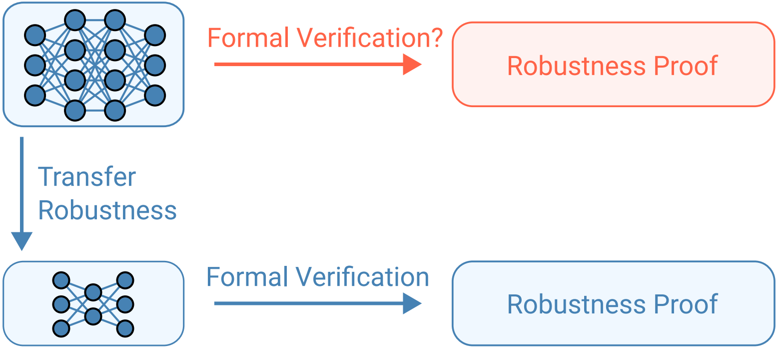

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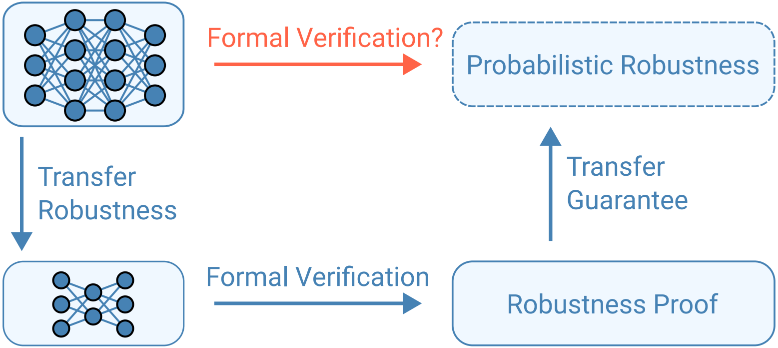

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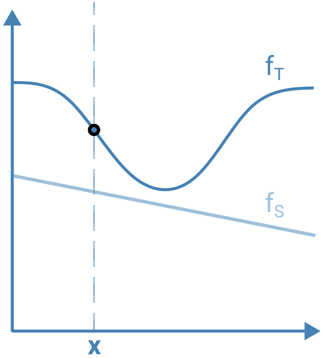

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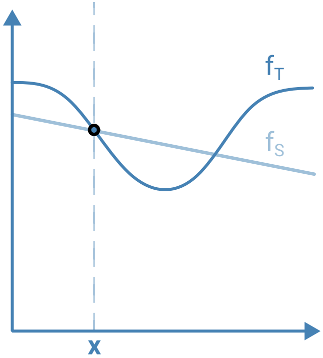

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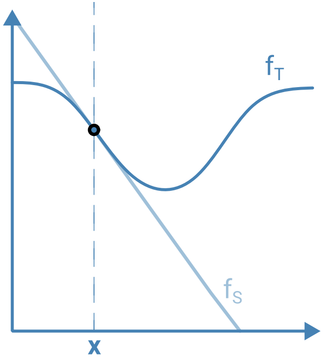

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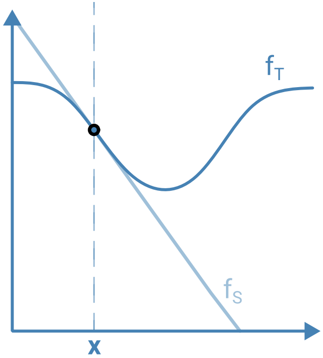

Where we will use $y = \textbf{class}(f_T(\mathbf{x}))$
Under *perfect conditions*, f_S is as robust as f_T
Assumes both functions are linear in a metric ball $B_r(\mathbf{x})$!

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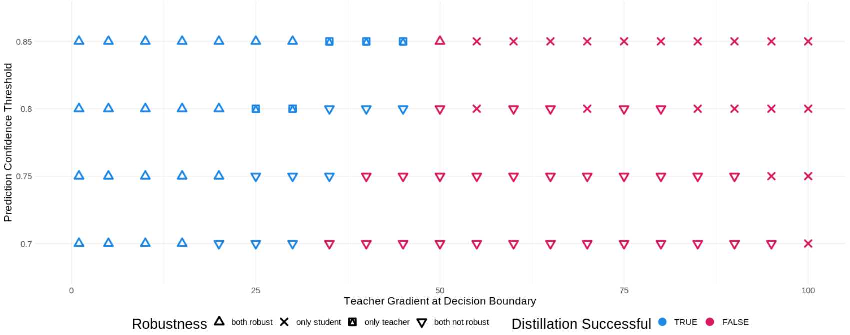
We sample sufficiently N iid from some dataset with additive noise

Does It Work? Experimental Results

We constructed f_T with known robustness properties and checked if robustness transferred through distillation



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  - If we consider balls of any size: we require local linearity at arbitrary scale
  - If we consider only small balls: maybe none are ϵ-likely (high dimensional data)

Property-Based Robustness Guarantees

Addressing issues: Local Robustness Checks

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For a classifier $f : \mathcal{X} \to \mathbb{R}^n$, a *robustness oracle* is defined as

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We can also use (non-exact) oracles that find a counterexample with attacks (e.g. PGD)

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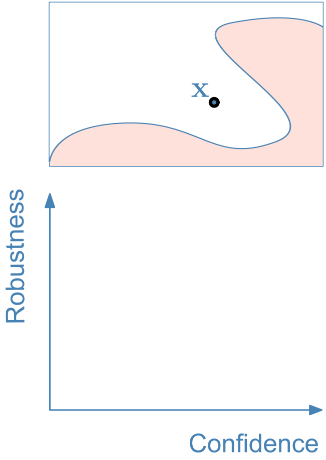

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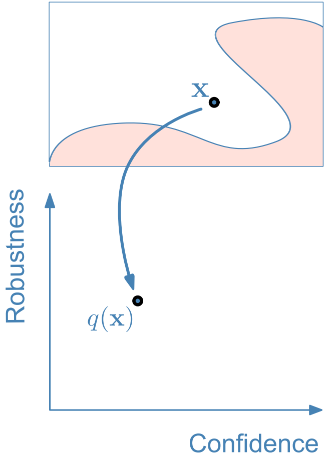

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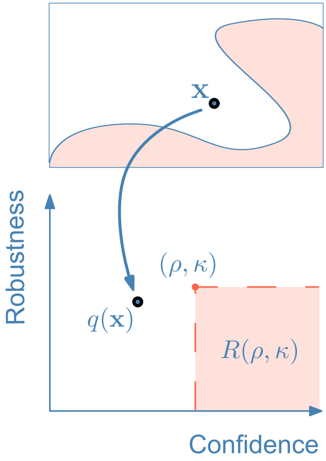
$$R(\rho, \kappa) = \{(\rho', \kappa') \in \mathbb{R}^2 : \rho' < \rho, \kappa' \geq \kappa\} \tag{14}$$



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Answer: *We can sample ϵ-nets in $\mathcal{Q}$!*

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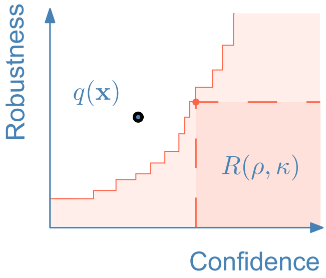

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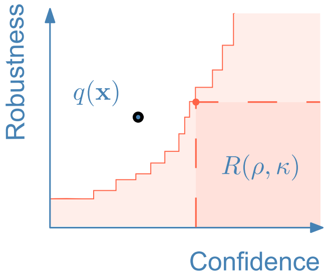
$|N| = s(\epsilon, \delta, d)$ depends *only on the VC-dimension d of $\mathcal{R}$, not on $\mathcal{X}$*



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We can use conditional guarantees to give "*customized*" robustness lower bounds for each prediction!

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We use *rank statistics* to estimate a bound from the sample!

Obtaining Constant Bounds with Rank Statistics

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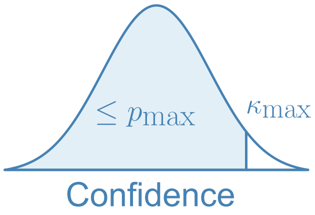

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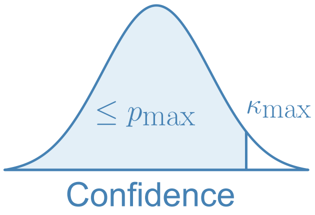

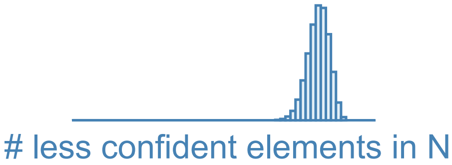

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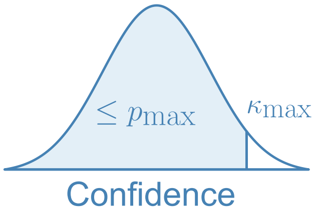

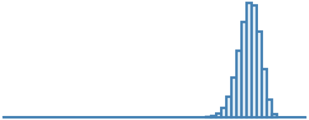

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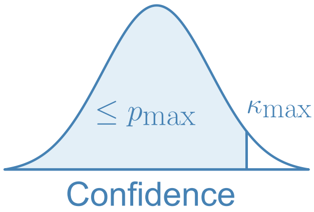

...and use *Chernoff bounds*

$$\Pr(\kappa_{\max} < N_{(i)}) < \delta \text{ s.t.} \tag{25}$$

$$i < c(|N|, p_{\max}, \delta) = \left\lfloor |N| p_{\max} - \sqrt{2|N| p_{\max} \ln\left(\frac{1}{\delta}\right)} \right\rfloor \tag{26}$$

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$$\forall \rho \forall \kappa \leq \kappa_{\max} : \{q(\mathbf{x}) : \mathbf{x} \in N\} \cap R(\rho, \kappa) = \emptyset \Rightarrow \Pr\left(\mathbf{rob}_f(X) < \rho \,|\, \mathbf{conf}_f(X) \geq \kappa\right) < \frac{\epsilon}{1 - p_{\max}} \tag{27}$$

If we have no counterexample in N, f *is probably approximately globally* (ρ, κ)*-robust*

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We expect for a given κ

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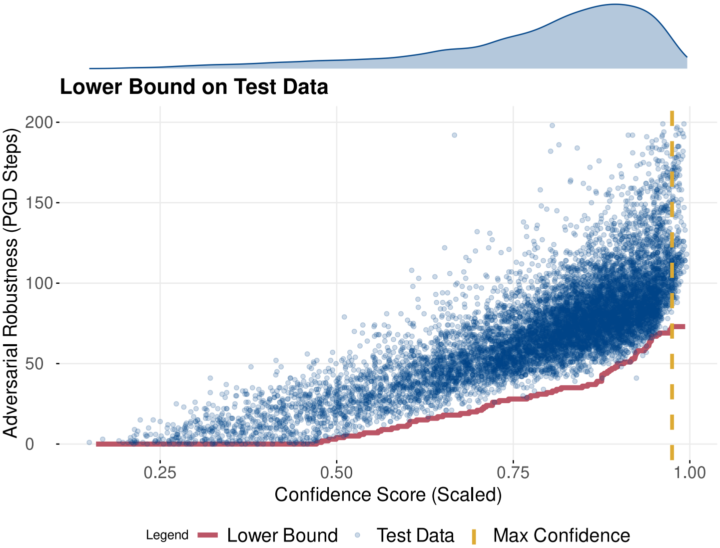

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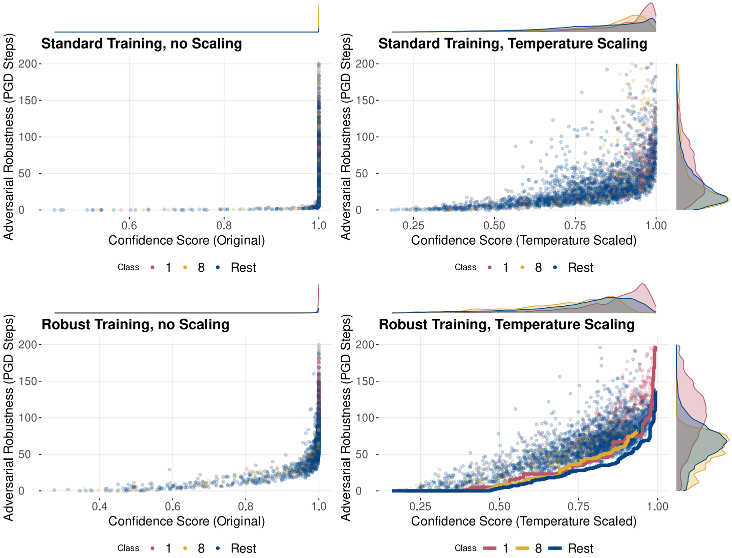

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This method also generalizes to learning other rules in black-box ML

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