AI CENTER FEE CTU

FACULTY OF ELECTRICAL ENGINEERING CTU IN PRAGUE

Combining Continuous and Combinatorial Optimization for Multi-goal Trajectory Planning

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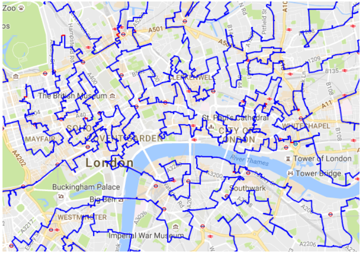
Traveling Salesman Problem (TSP)

Problem 1 TSP

Given a set of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city.

- **Exact solutions** ‘
  - **Concorde** `math.uwaterloo.ca/tsp/concorde.html` (Integer Linear Programming (ILP))
- **Heuristic algorithms** ‘
  - **LKH** – K. Helsgaun efficient implementation of the Lin-Kernighan heuristic (1998). `http://www.akira.ruc.dk/~keld/research/LKH/`

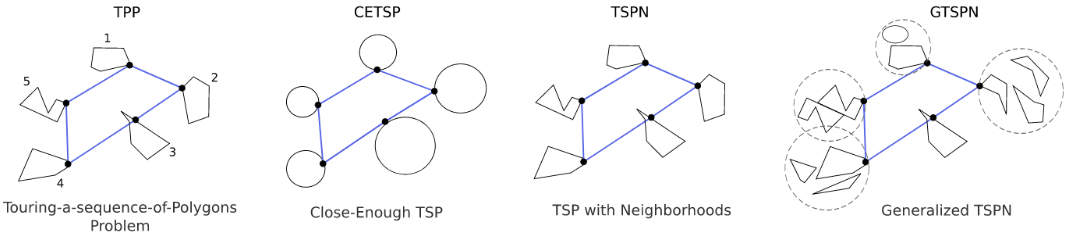


`https://www.math.uwaterloo.ca/tsp/pubs/`

Multi-Goal Planning

Problem 2 Multi-Goal Planning

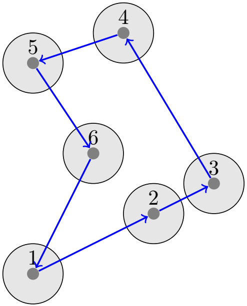
Having a **set of locations or neighborhoods** to be visited, determine the cost-efficient path or trajectory to visit them.



Alatartsev, S., Stellmacher, S., Ortmeier, F. (2015): **Robotic Task Sequencing Problem: A Survey**. Journal of Intelligent & Robotic Systems.

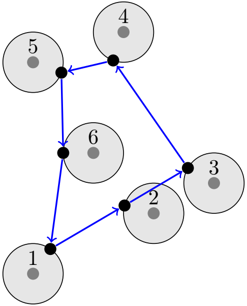
Decoupled Solution of Multi-Goal Planning

First, determine the sequence.



A solution of the TSP for the centers of the disks

Second, solve the Touring problem.

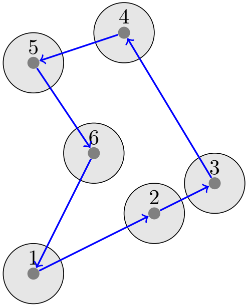


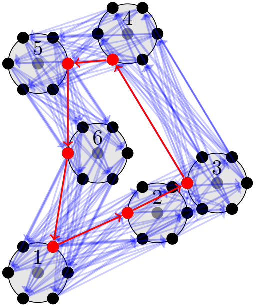
A solution of the **CETSP**

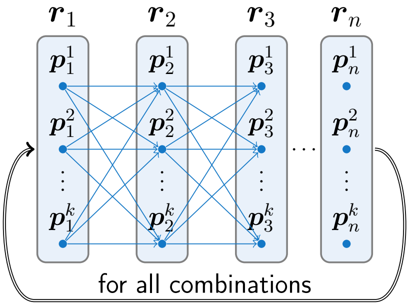
Sampling-based Solution of the Touring problem

- Sample each region (neighborhood) with k samples, e.g., $k = 6$.
- Construct graph and find the shortest tour in by graph search in $\mathcal{O}(nk^3)$ for n regions and nk^2 edges in the sequence.

For the closed path, we need to examine all k possible starting locations.

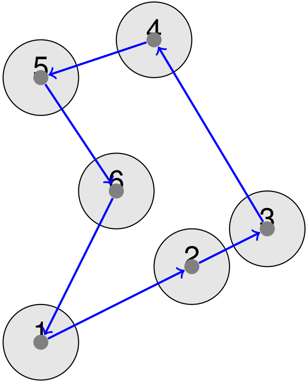


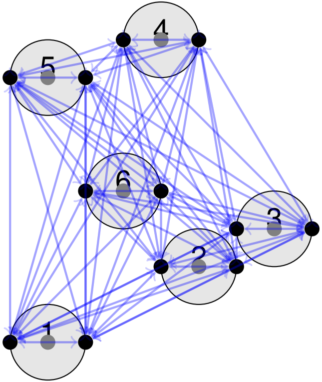




Sampling-based Solution of the TSPN

- For an unknown sequence of the visits to the regions, there are $\mathcal{O}(n^2k^2)$ possible edges.
- Finding the shortest path is NP-hard, as it can be formulated as the Generalized TSP.





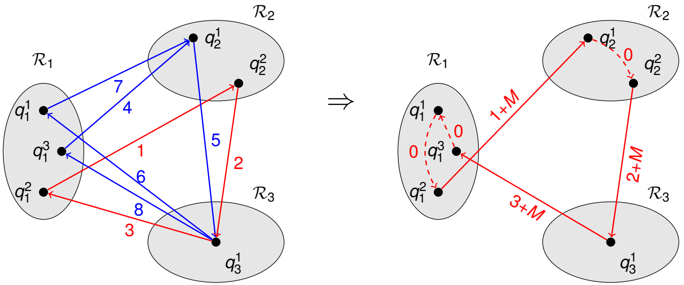
Noon-Bean transformation (GATSP to ATSP)

1. Create a zero-length cycle in each set and set all other arcs to ∞ (or 2M).

   To ensure all vertices of the cluster are visited before leaving the cluster.

2. For each edge (q_i^m, q_j^n) create an edge (q_i^m, q_j^{n+1}) with a value increased by large M.

   To ensure visit of all vertices in a cluster before the next cluster.



Planning with Curvature-constrained Paths

General aviation

Unmanned vehicles

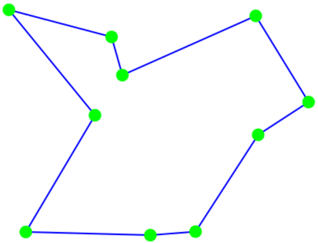
Flying cars

Dubins Traveling Salesman Problem

- Visit the given set of locations.
- Collect required data at the locations.
- Consider a fixed-wing aerial vehicle.
- Exploit the **Dubins vehicle** model
  - Minimal turning radius ρ.
  - Constant forward velocity v.
  - State of the vehicle is $q = (x, y, \theta)$.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = v \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{u}{\rho} \end{bmatrix}, \quad |u| \le 1, \quad (1)$$

Traveling Salesmen Problem (TSP)



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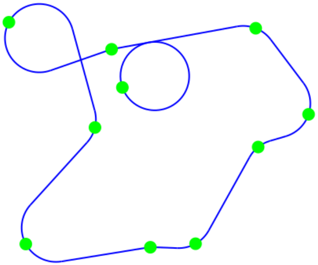
Dubins TSP (DTSP)



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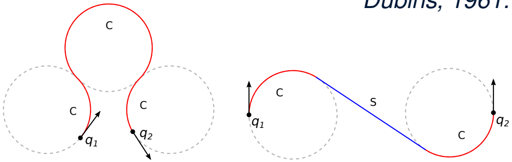
Dubins, 1961.



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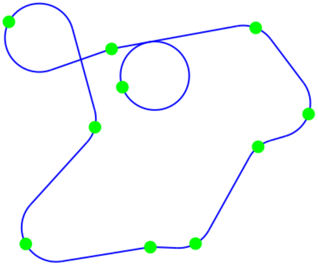

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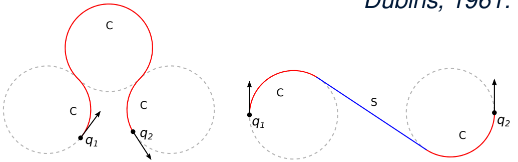

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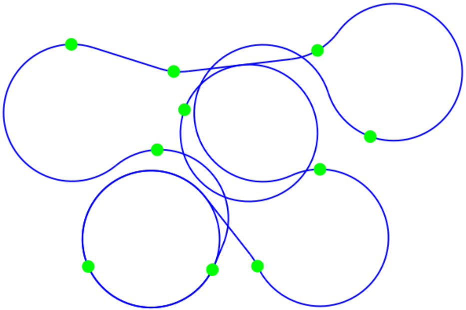
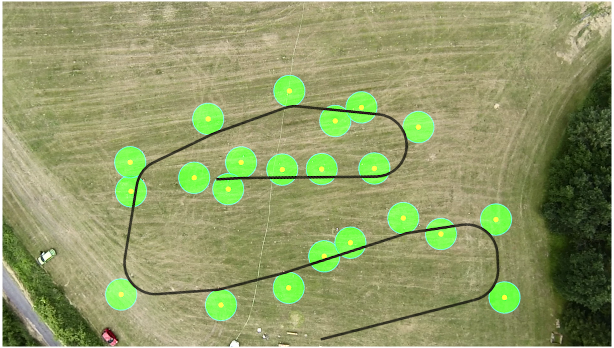
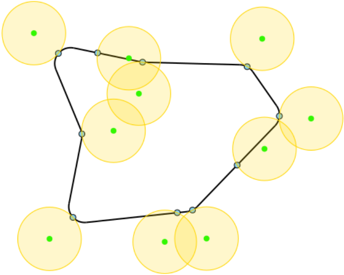


Dubins Traveling Salesman Problem with Neighborhoods

- Utilizes non-zero sensing radius of the sensor.
- Decreases length of the tour.
- Makes the problem more challenging.

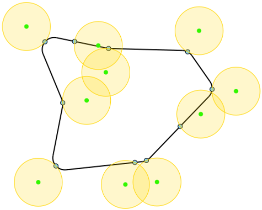


DTSPN



Existing Approaches to the DTSP(N)

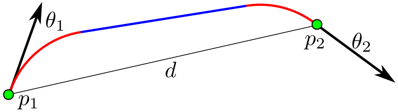
- Heuristic (decoupled & evolutionary) approaches
  - *Savla et al., 2005*
  - *Ma and Castanon, 2006*
  - *Macharet et al., 2011*
  - *Macharet et al., 2012*
  - *Ny et al., 2012*
  - *Yu and Hang, 2012*
  - *Macharet et al., 2013*
  - *Zhant et al., 2014*
  - *Macharet and Campost, 2014*
  - *Váňa and Faigl, 2015*
  - *Isaiah and Shima, 2015*
  - ...

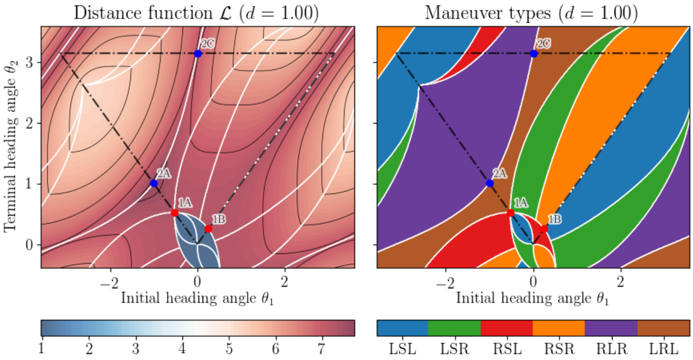


- Sampling-based approaches
  - *Obermeyer, 2009*
  - *Oberlin et al., 2010*
  - *Macharet et al., 2016*
- Convex optimization
  - (Only if the locations are far enough)
  - *Goac et al., 2013*
- **Lower bound for the DTSP**
  - Dubins Interval Problem (DIP)
  - *Manyam et al., 2016*
  - DIP-based inform sampling
  - *Váňa and Faigl, 2017*
- **Lower bound for the DTSPN**
  - Using Generalized DIP (GDIP)
  - *Váňa and Faigl, 2018, 2020, 2022 (In review)*

Properties of the Dubins distance function

- Piecewise-continuous function.
- Closed form expression.
- Fast to compute 0.5μs.
- Continuous for $d > 4$,
  where $d = \frac{\|p_2 - p_1\|}{\rho}$.
- Normalized form
  - $q_1 = (p_1, \theta_1) = (0, 0, \theta_1)$,
  - $q_2 = (p_2, \theta_2) = (d\rho, 0, \theta_2)$.





Dubins Interval Problem (DIP)

- Determine the shortest Dubins maneuver connecting p_1 and p_2 given the angle intervals $\theta_1 \in [\theta_1^{min}, \theta_1^{max}]$ and $\theta_2 \in [\theta_2^{min}, \theta_2^{max}]$. (closed-form solution)

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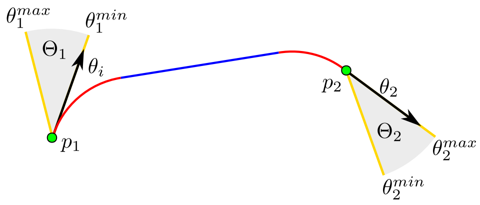

Manyam, Rathinam, and Casbeer, 2016

Case	Maneuvers		Conditions on θ_1 and θ_2
1)	S or L_ψ or R_ψ [1]		
2)	LS or LR_ψ	for	$\theta_1 = \theta_1^{max}$ and $\theta_2 \in \Theta_2$
3)	RS or RL_ψ	for	$\theta_1 = \theta_1^{min}$ and $\theta_2 \in \Theta_2$
4)	SL or $R_\psi L$	for	$\theta_1 \in \Theta_1$ and $\theta_2 = \theta_2^{min}$
5)	SR or $L_\psi R$	for	$\theta_1 \in \Theta_1$ and $\theta_2 = \theta_2^{max}$
6)	LSR	for	$\theta_1 = \theta_1^{max}$ and $\theta_2 = \theta_2^{max}$
7)	LSL or $LR_\psi L$	for	$\theta_1 = \theta_1^{max}$ and $\theta_2 = \theta_2^{min}$
8)	RSL	for	$\theta_1 = \theta_1^{min}$ and $\theta_2 = \theta_2^{min}$
9)	RSR or $RL_\psi R$	for	$\theta_1 = \theta_1^{min}$ and $\theta_2 = \theta_2^{max}$

Satyanarayana G Manyam, Sivakumar Rathinam, David Casbeer, and Eloy Garcia. Tightly bounding the shortest dubins paths through a sequence of points. *Journal of Intelligent & Robotic Systems*, 88(2):495–511, 2017.

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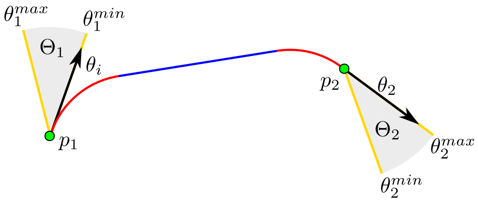
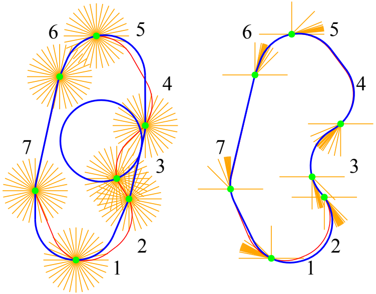


Manyam, Rathinam, and Casbeer, 2016

$\Rightarrow$

Dubins Touring Problem (DTP)

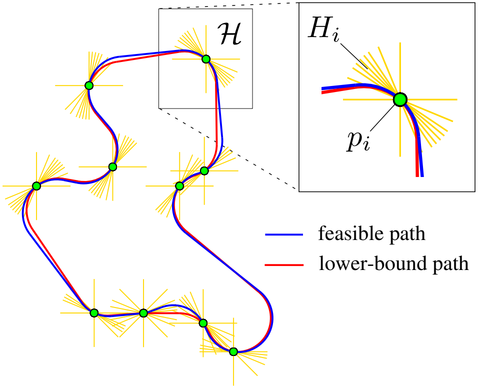


Jan Faigl, Petr Váňa, Martin Saska, Tomáš Báča, and Vojtěch Spurný. On solution of the dubins touring problem. In *European Conf. on Mobile Robots (ECMR)*, pages 1–6. IEEE, 2017.

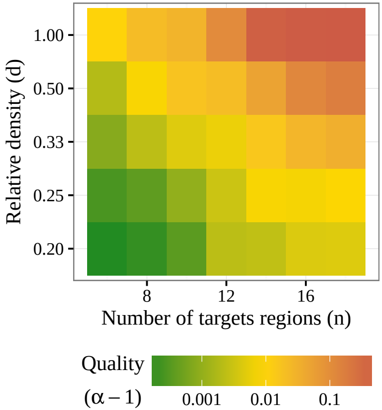
First attempt to solve DTSP optimally (2016)

- Find the optimum without a priory known sequence using Noon-Bean transformation.

Dubins TSP (unknown sequence)

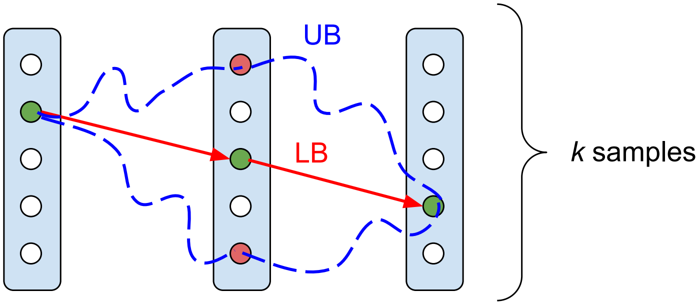


Quality of the solution found in 60s



How to remove (bound) intervals?

- Remove heading angle intervals which cannot contribute to the optimum.
- Testing one location takes $\mathcal{O}(k^3)$.



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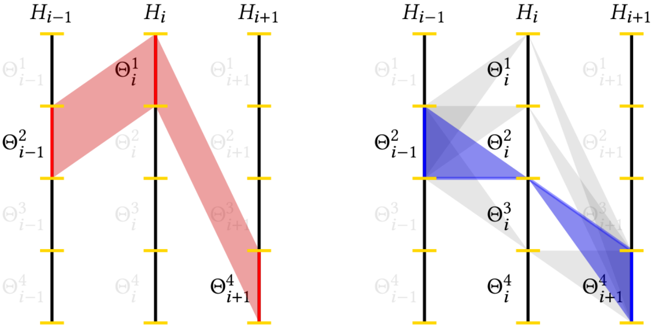

(a) Lower bound for $w = 1$

(b) Upper bound for $w = 1$

- $\mathcal{L}_L$ - Lower bound.
- $\mathcal{L}_U$ - Upper bound.

Condition 1 for NOT removing interval Θ_i

$$\exists\, \Theta_{i-w} \in H_{i-w}, \exists\, \Theta_{i+w} \in H_{i+w} : \mathcal{L}_L(\Theta_{i-w}, \Theta_i) + \mathcal{L}_L(\Theta_{i-w}, \Theta_i) \leq \mathcal{L}_U(\Theta_{i-w}, \Theta_{i+w}).$$

Maximization Dubins Interval Problem (Max-DIP)

- Determine the **longest** Dubins maneuver connecting p_i and p_j given the angle intervals $\theta_i \in [\theta_i^{min}, \theta_i^{max}]$ and $\theta_j \in [\theta_j^{min}, \theta_j^{max}]$.
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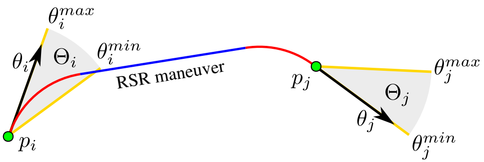
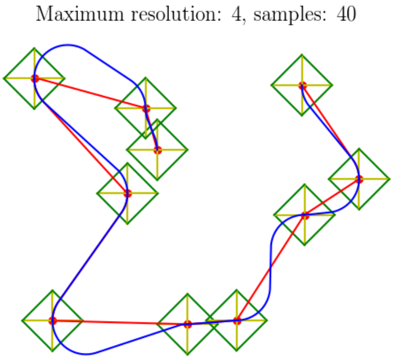
Max-DIP



Dubins Touring Problem (DTP)



Petr Váňa and Jan Faigl. Bounding optimal headings in the dubins touring problem. In *Proceedings of the 37th ACM/SIGAPP Symposium on Applied Computing*, pages 770–773, 2022.

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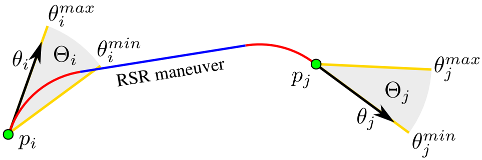

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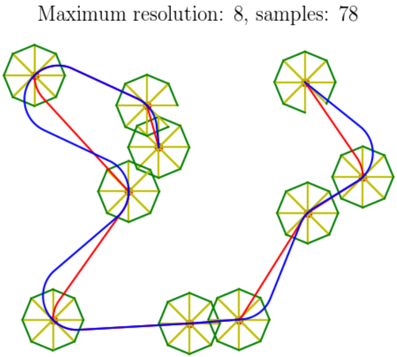

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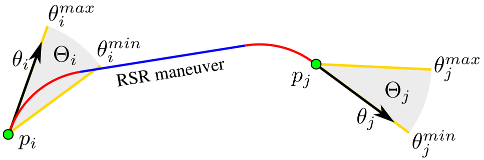

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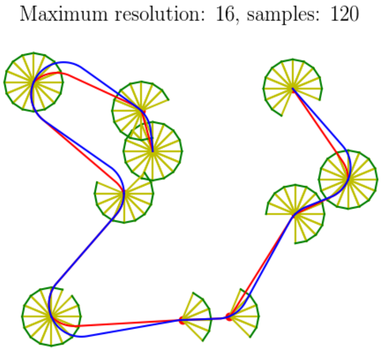

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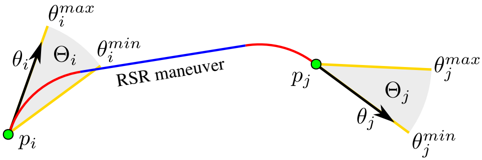

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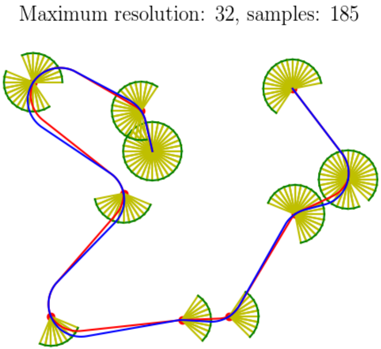

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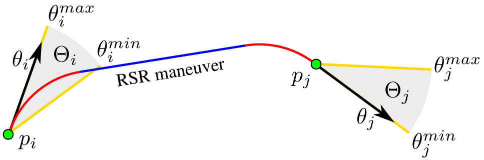

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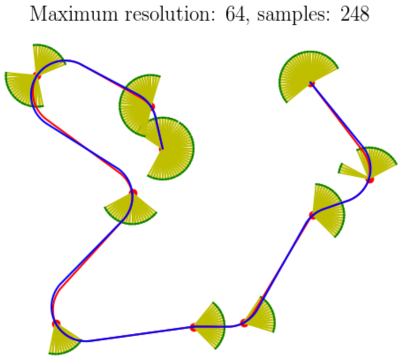

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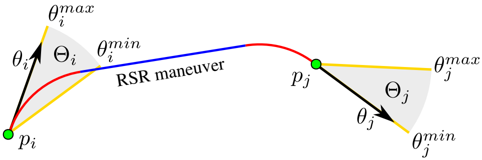

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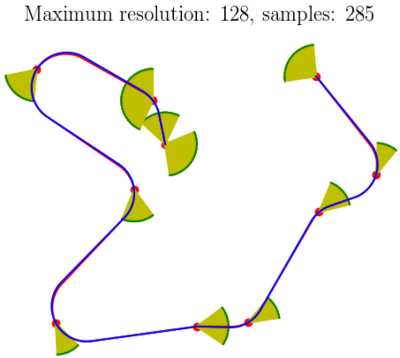

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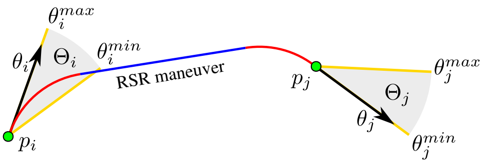
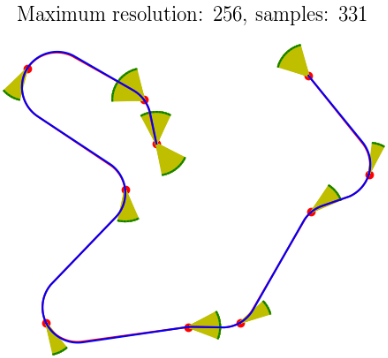


Dubins Touring Problem (DTP)

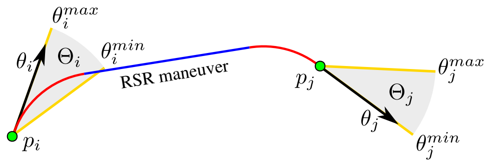


Petr Váňa and Jan Faigl. Bounding optimal headings in the dubins touring problem. In *Proceedings of the 37th ACM/SIGAPP Symposium on Applied Computing*, pages 770–773, 2022.

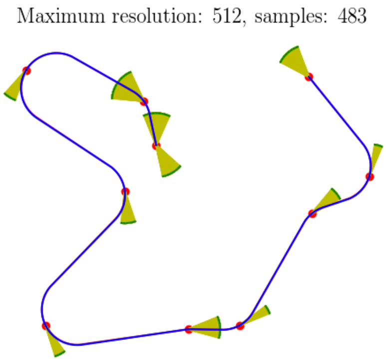
Maximization Dubins Interval Problem (Max-DIP)

- Determine the **longest** Dubins maneuver connecting p_i and p_j given the angle intervals $\theta_i \in [\theta_i^{min}, \theta_i^{max}]$ and $\theta_j \in [\theta_j^{min}, \theta_j^{max}]$.
- Remove heading angle intervals which cannot contribute to the optimum.

Max-DIP



Dubins Touring Problem (DTP)



Petr Váňa and Jan Faigl. Bounding optimal headings in the dubins touring problem. In *Proceedings of the 37th ACM/SIGAPP Symposium on Applied Computing*, pages 770–773, 2022.

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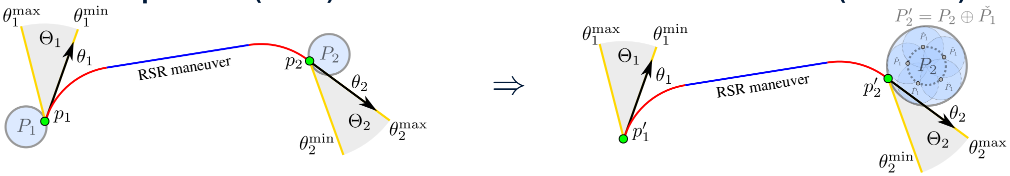
Generalized Dubins Interval Problem (GDIP)

- Determine the **shortest** Dubins maneuver connecting P_1 and P_2 given the angle intervals $\theta_1 \in [\theta_1^{min}, \theta_1^{max}]$ and $\theta_2 \in [\theta_2^{min}, \theta_2^{max}]$

Full problem (GDIP) ⇒ **One-side version (OS-GDIP)**



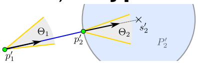
- Transformation from the GDIP to the OS-GDIP:
  - $P_1' = \{p_1'\} = \{(0,0)\}$
  - $P_2' = P_2 \oplus \check{P}_1 = \cup\{p_b - p_a, p_a \in P_1, p_b \in P_2\}$

Petr Váňa and Jan Faigl. Optimal Solution of the Generalized Dubins Interval Problem. In *Robotics: Science and Systems (RSS)*, 2018. Best student paper award nominee.

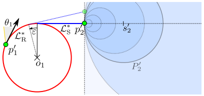
Optimal Solution of the GDIP

Closed-form expressions (1-6)

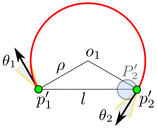
1) **S type**



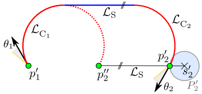
2) **CS type**



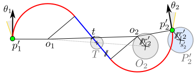
3) **C$_\psi$ type**



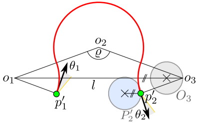
4) **CSC type**



5) **CS$\overline{\text{C}}$ type**

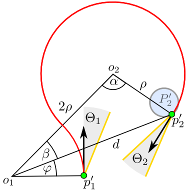


6) **C$\overline{\text{C}}_\psi$C type**



Convex optimization (7)

7) **C$\overline{\text{C}}_\psi$ type**

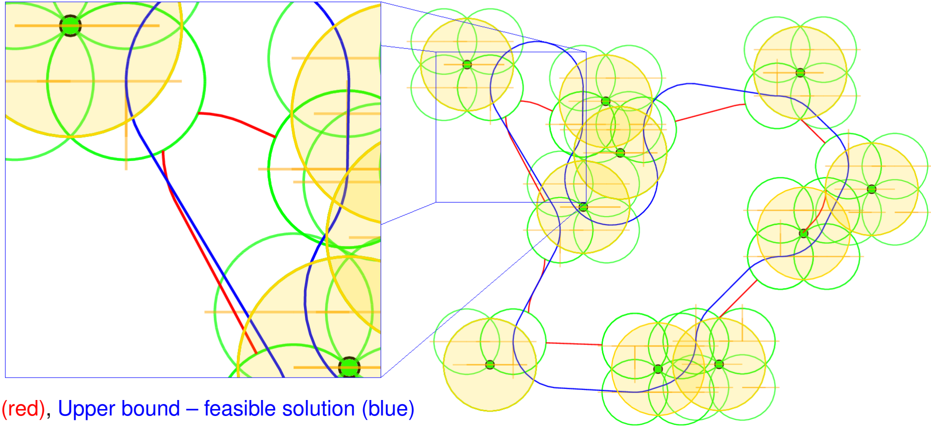


Average computational time

Problem	Time [μs]	Ratio
Dubins maneuver	0.58	1.00
DIP	2.86	4.93
GDIP	12.63	21.78

Computing bounds for a single sequence of the DTSPN

Resolution: 4 Gap: 69.3 % Time: 0.079 s

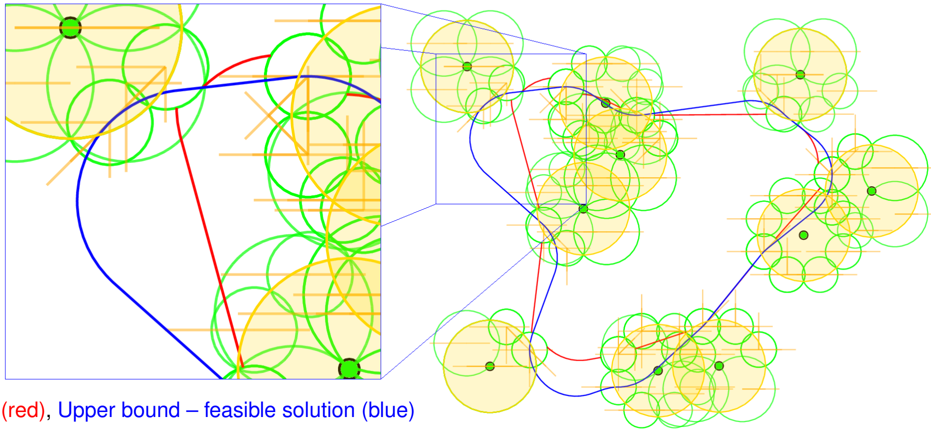


Fixed sequence!

Lower-bound (red), Upper bound – feasible solution (blue)

Computing bounds for a single sequence of the DTSPN

Resolution: 8 Gap: 39.4 % Time: 0.211 s

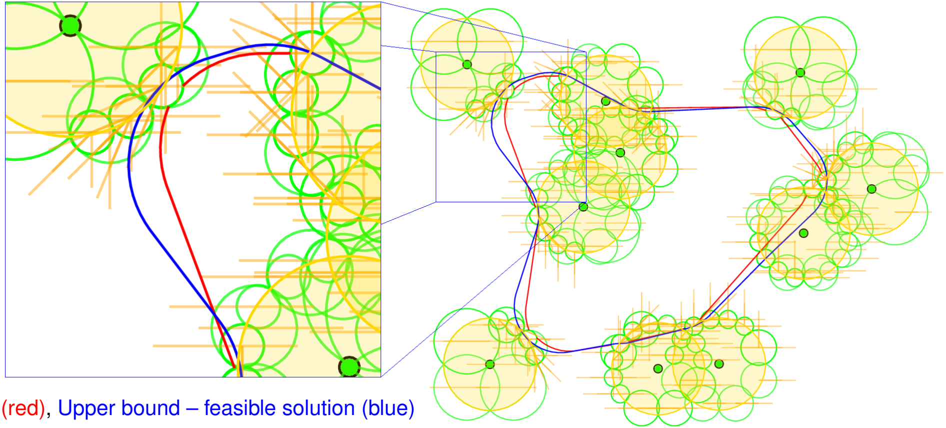


Fixed sequence!

Lower-bound (red), Upper bound – feasible solution (blue)

Computing bounds for a single sequence of the DTSPN

Resolution: 16 Gap: 19.9 % Time: 0.552 s

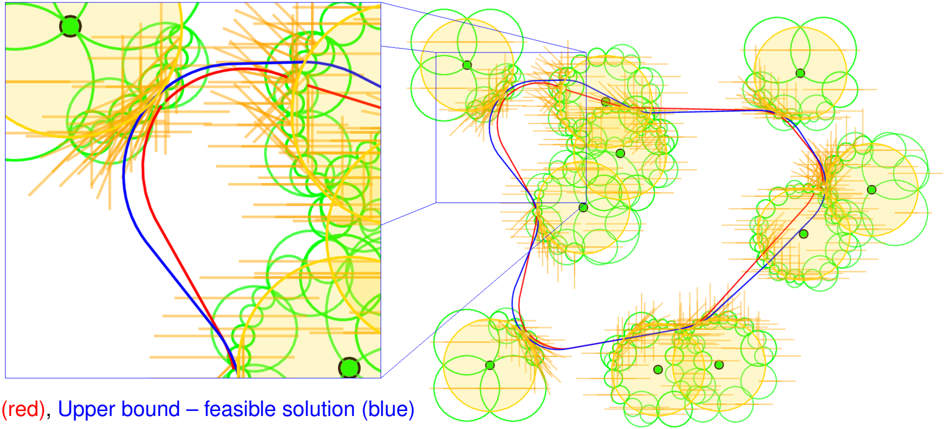


Fixed sequence!

Lower-bound (red), Upper bound – feasible solution (blue)

Computing bounds for a single sequence of the DTSPN

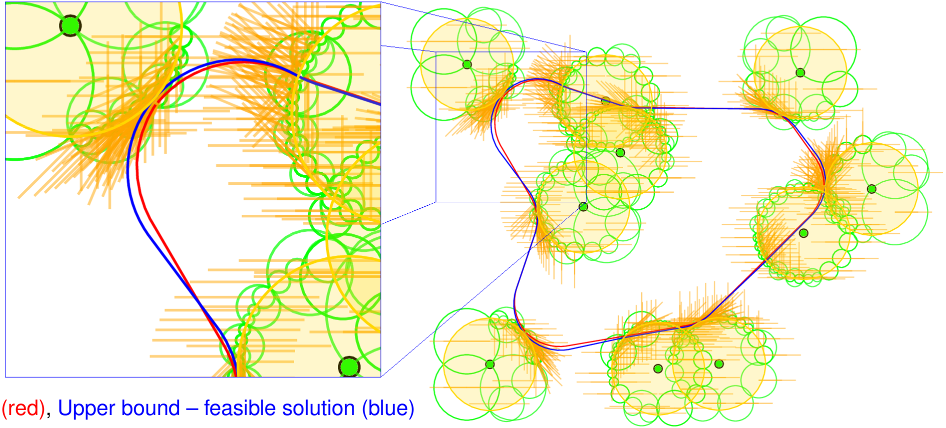
Resolution: 32 Gap: 10.7 % Time: 1.292 s



Lower-bound (red), Upper bound – feasible solution (blue)

Computing bounds for a single sequence of the DTSPN

Resolution: 64 Gap: 5.3 % Time: 3.183 s

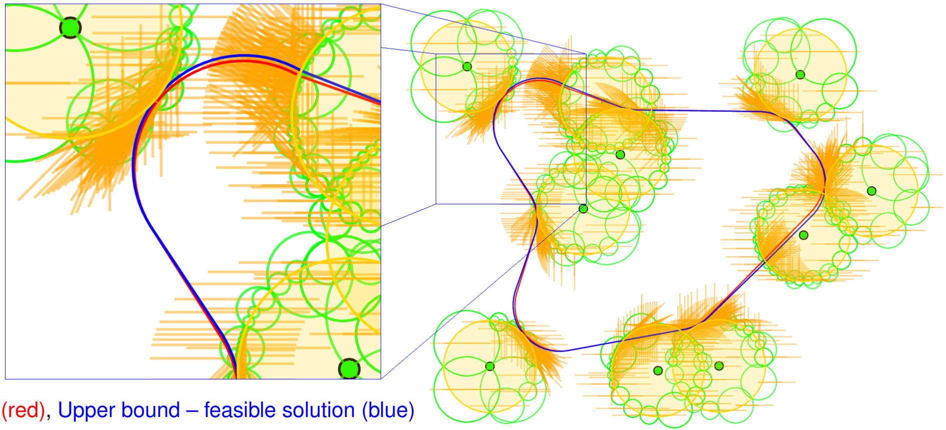


Fixed sequence!

Lower-bound (red), Upper bound – feasible solution (blue)

Computing bounds for a single sequence of the DTSPN

Resolution: 128   Gap: 2.6 %   Time: 8.994 s

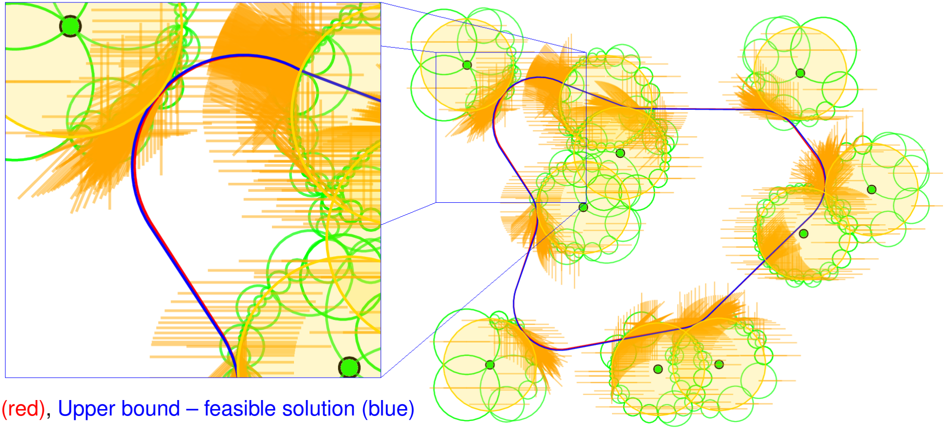


Fixed sequence!

Lower-bound (red), Upper bound – feasible solution (blue)

Computing bounds for a single sequence of the DTSPN

Resolution: 256   Gap: 1.3 %   Time: 33.474 s

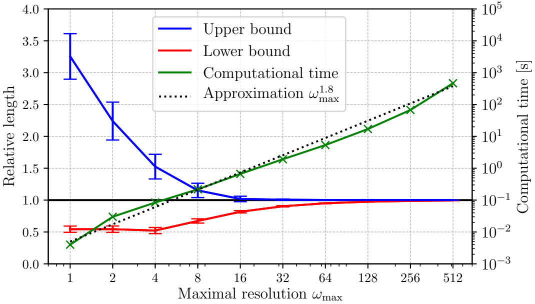


Fixed sequence!

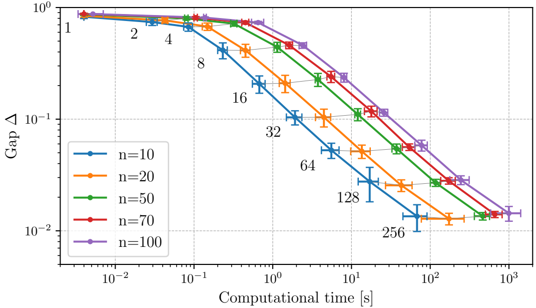
Lower-bound (red), Upper bound – feasible solution (blue)

Convergence for a single sequence of the DTSPN

Convergence for 10 regions

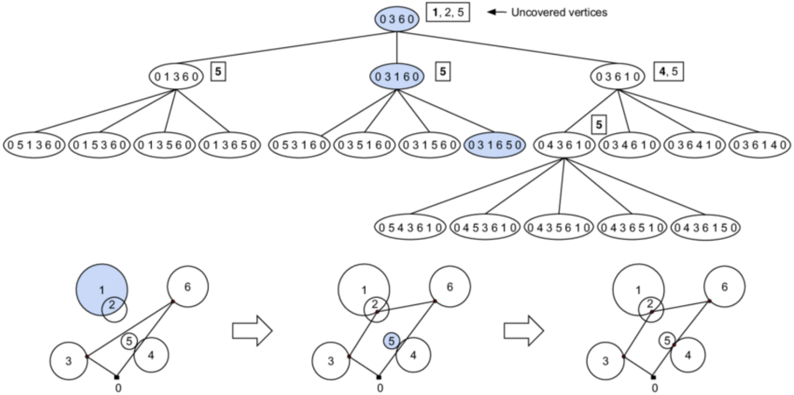


Various number of regions



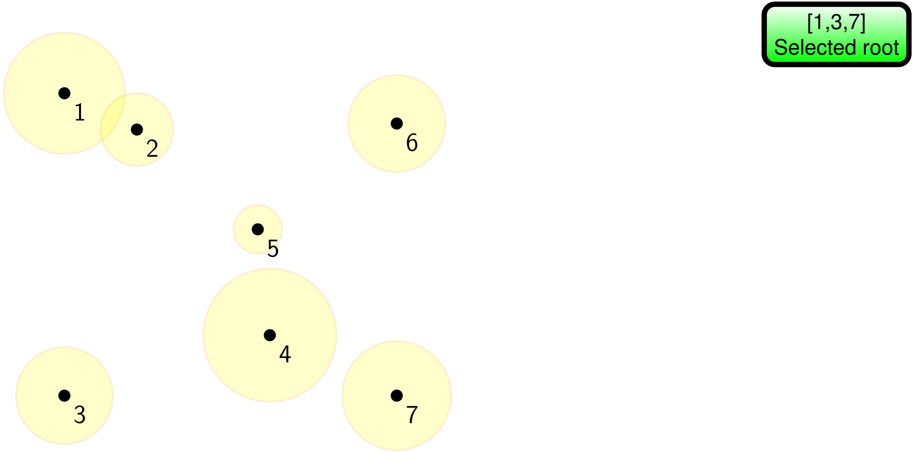
- The computational time can be approximated by $\mathcal{O}(n\omega_{max}^{1.8})$ where ω_{max} is maximal resolution.

Branch-and-Bound (BNB) framework

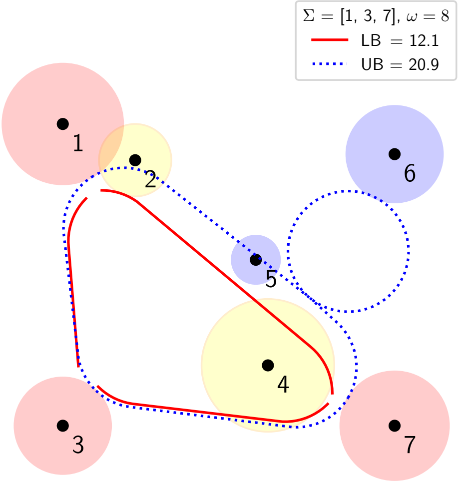


Walton Pereira Coutinho, Roberto Quirino do Nascimento, Artur Alves Pessoa, and Anand Subramanian. A branch-and-bound algorithm for the close-enough traveling salesman problem. *INFORMS Journal on Computing*, 28(4):752–765, 2016.

Proposed Branch-and-Bound (BNB) algorithm

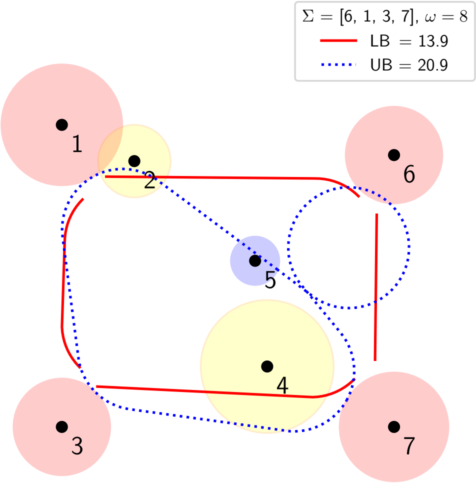


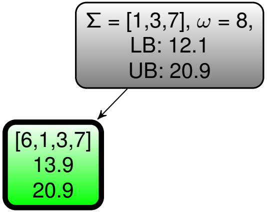
Proposed Branch-and-Bound (BNB) algorithm



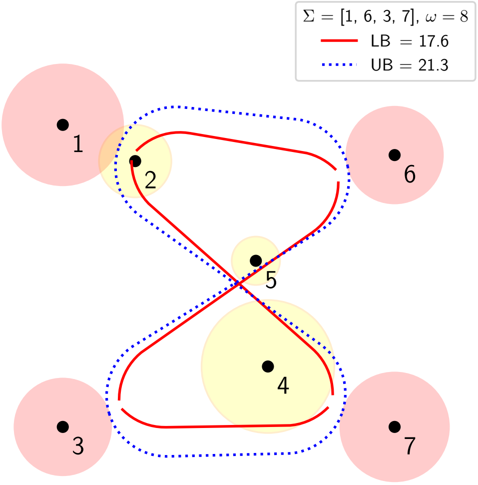
Σ = [1,3,7], ω = 8,
LB: 12.1
UB: 20.9

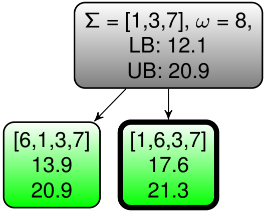
Proposed Branch-and-Bound (BNB) algorithm



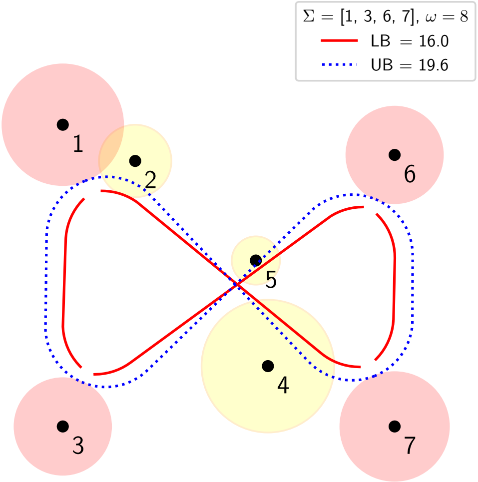


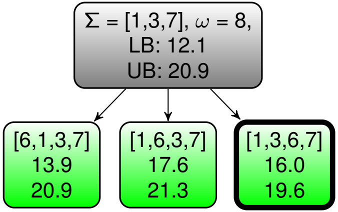
Proposed Branch-and-Bound (BNB) algorithm





Proposed Branch-and-Bound (BNB) algorithm





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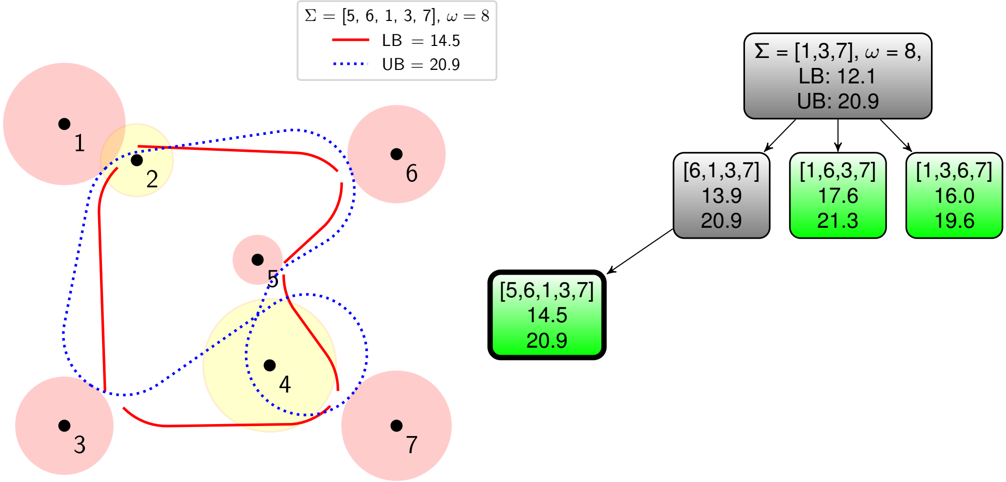

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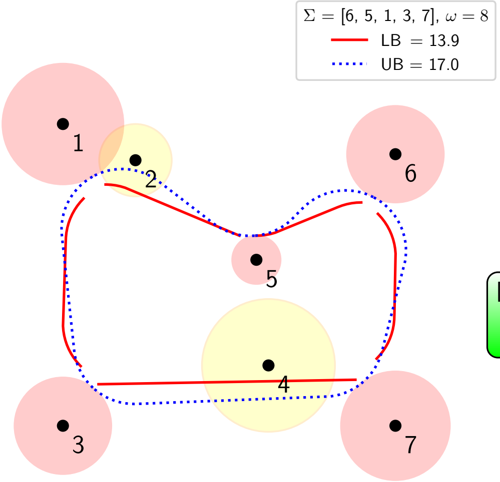

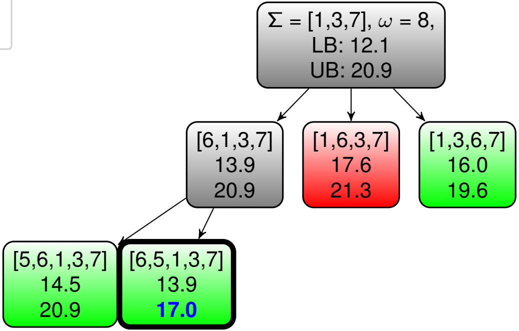

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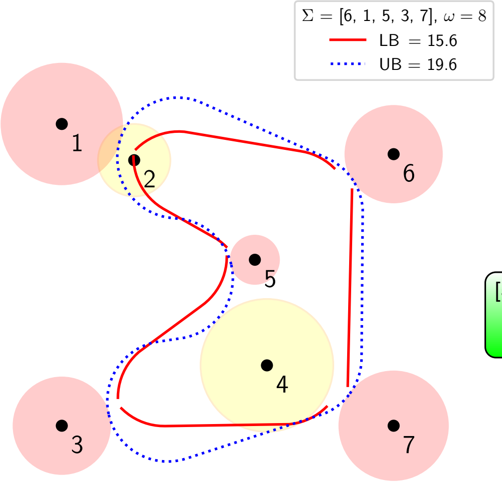

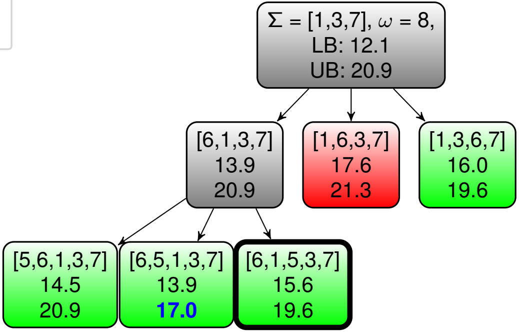

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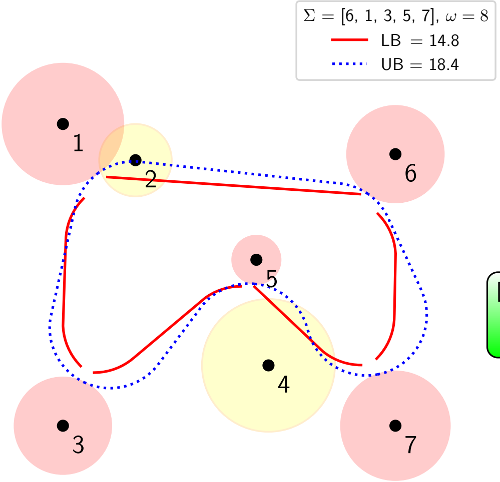

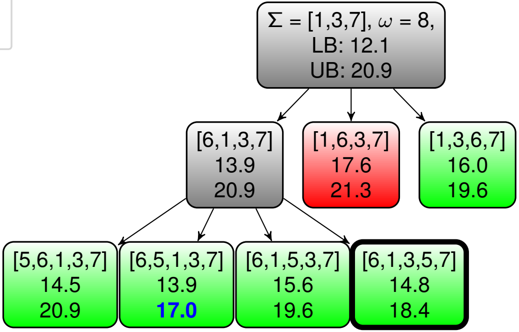

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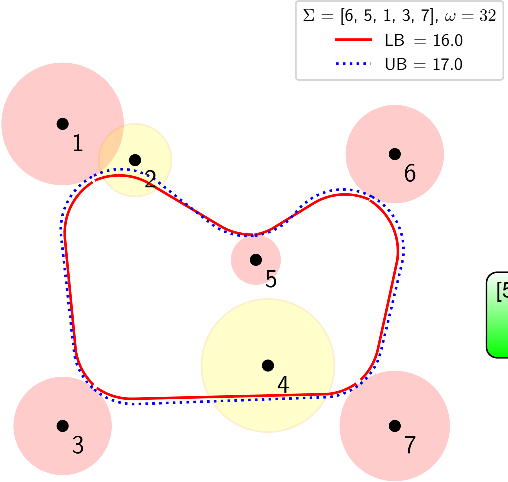

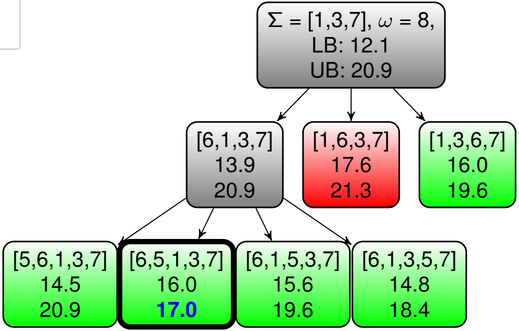

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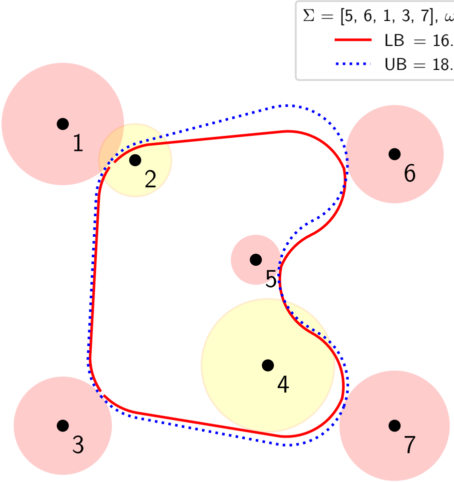

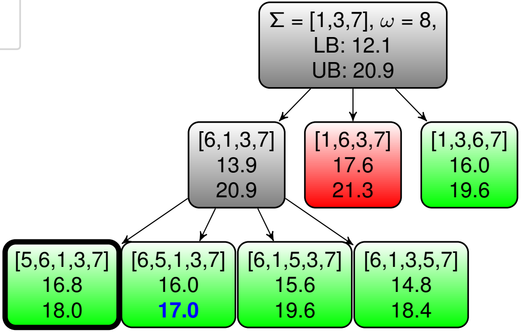

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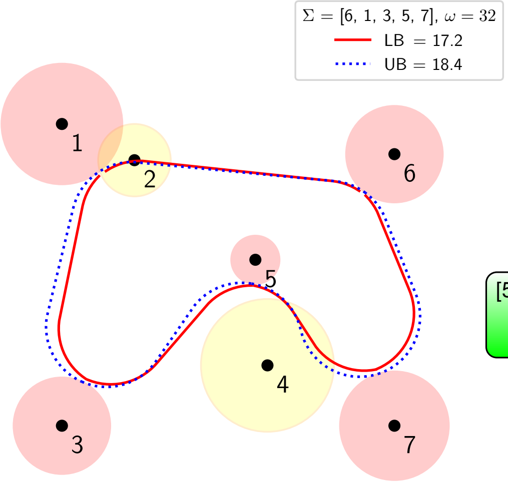
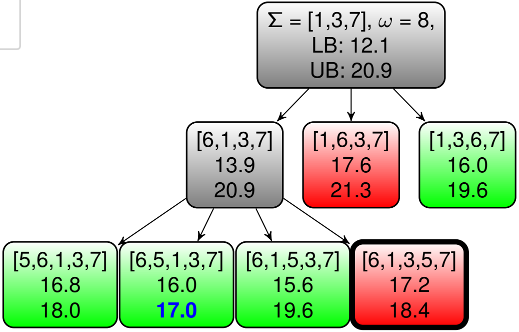
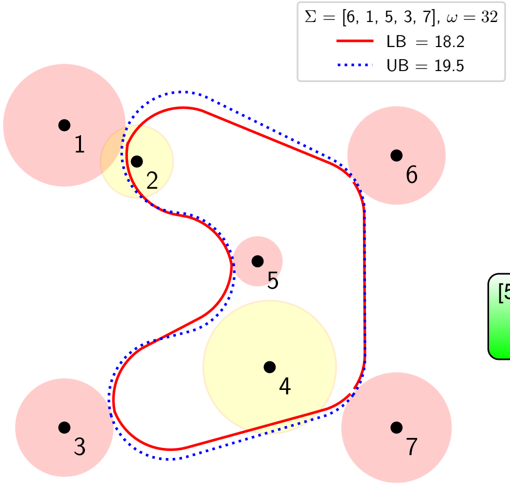
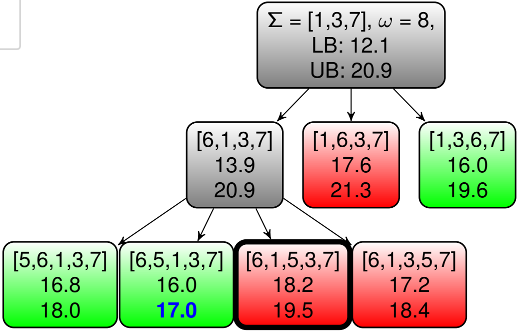




Proposed Branch-and-Bound (BNB) algorithm





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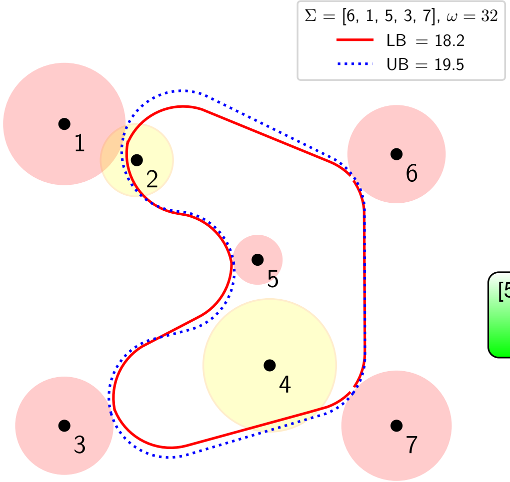

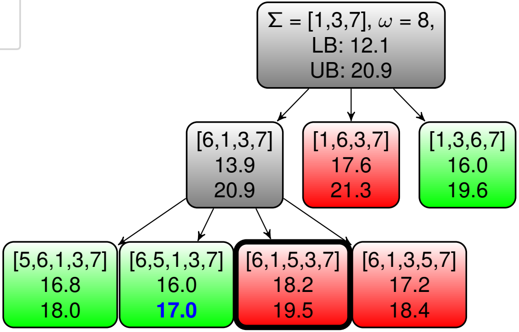

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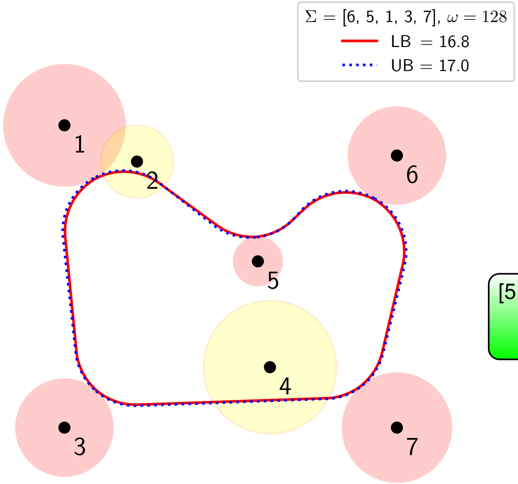

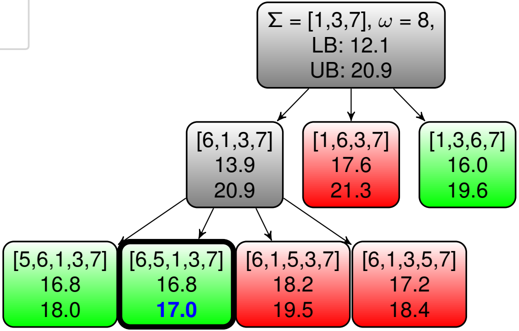

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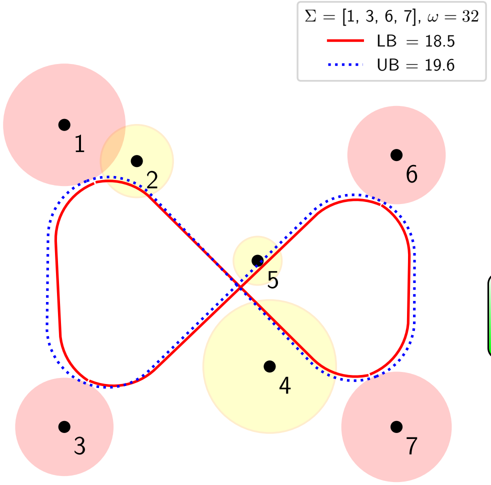

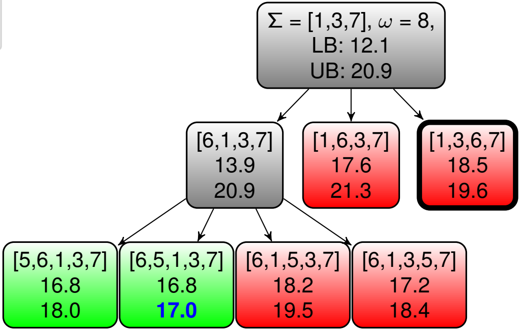

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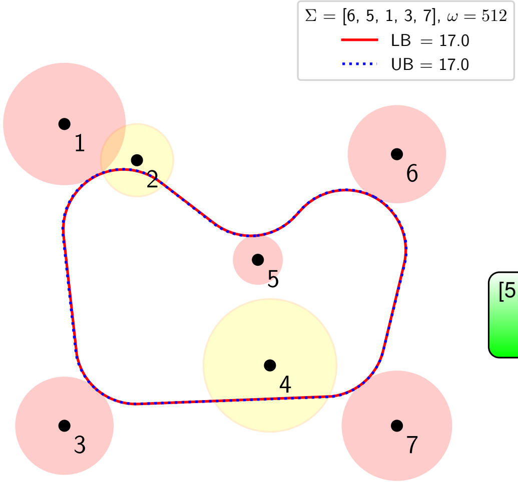

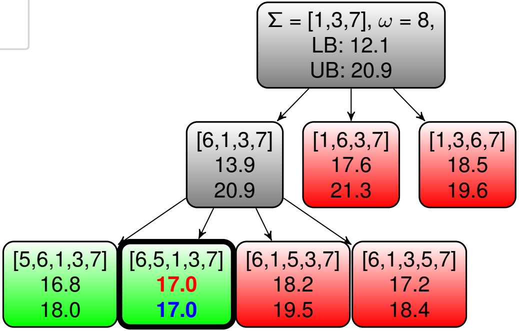

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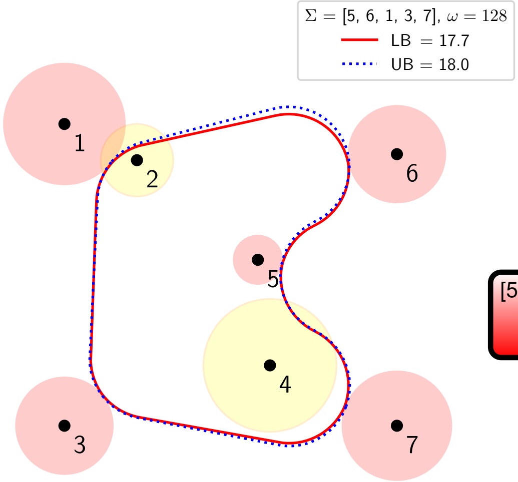

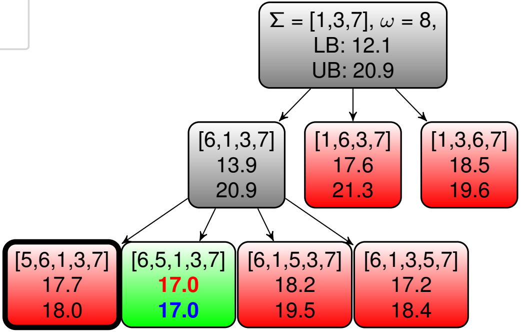

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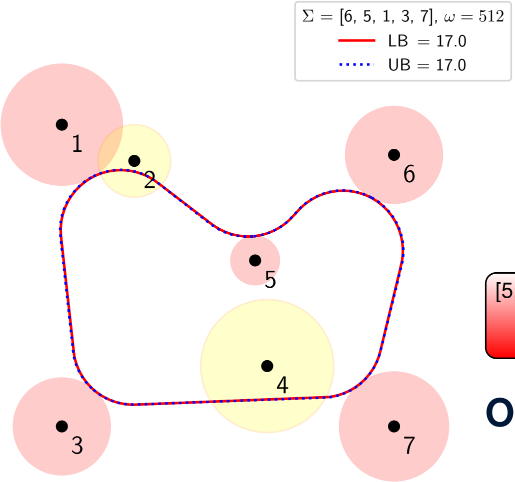
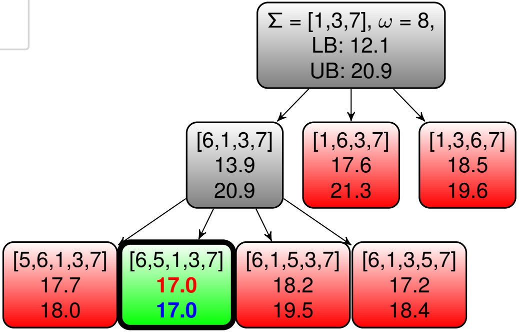
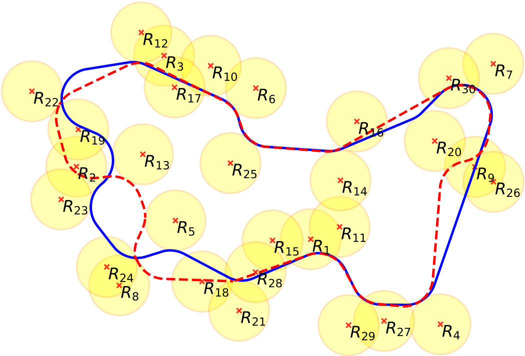




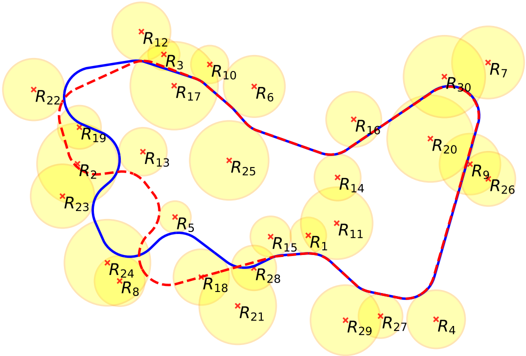
Optimal sequence found!

Example solutions for the DTSPN

Uniform sensing radius

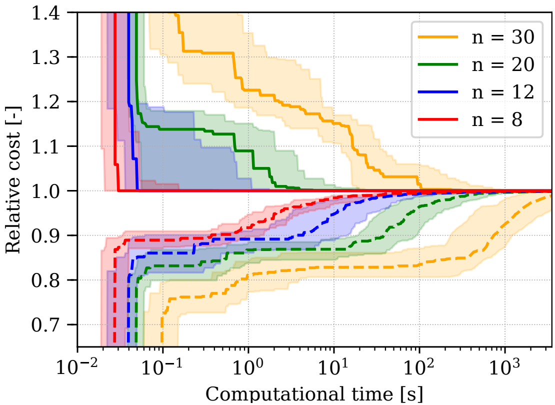


Various sensing radius



Summary and empirical results

- Proposed BNB for the DTSPN
  - Continuous part → GDIP.
  - Sequencing part → branching.
  - Sub-sequences bounded by LB/UB.
  - Neighborhoods → faster solutions.
- BNB algorithm implemented in Julia.
- Optimal GDIP solution in **C++11**.



https://github.com/comrob/OptimalDTSPN

Than you for your attention!

- Satyanarayana G Manyam, Sivakumar Rathinam, David Casbeer, and Eloy Garcia. Tightly bounding the shortest dubins paths through a sequence of points. *Journal of Intelligent & Robotic Systems*, 88(2):495–511, 2017.
- Jan Faigl, Petr Váňa, Martin Saska, Tomáš Báča, and Vojtěch Spurný. On solution of the dubins touring problem. In *European Conf. on Mobile Robots (ECMR)*, pages 1–6. IEEE, 2017.
- Petr Váňa and Jan Faigl. Bounding optimal headings in the dubins touring problem. In *Proceedings of the 37th ACM/SIGAPP Symposium on Applied Computing*, pages 770–773, 2022.
- Petr Váňa and Jan Faigl. Optimal Solution of the Generalized Dubins Interval Problem. In *Robotics: Science and Systems (RSS)*, 2018. Best student paper award nominee.
- Walton Pereira Coutinho, Roberto Quirino do Nascimento, Artur Alves Pessoa, and Anand Subramanian. A branch-and-bound algorithm for the close-enough traveling salesman problem. *INFORMS Journal on Computing*, 28(4):752–765, 2016.